Supplementary Material

A Two-Stage Stochastic Fuzzy Mixed-Integer Linear Programming Approach for Water Resource Allocation under Uncertainty in Ajabshir Qaleh Chay Dam

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A.1. Inexact fuzzy interval linear programming

FMP is an efficient method to deal with uncertainty in most of the real-world problems in the existence of ambiguous information. In this section, we introduce an inexact fuzzy interval linear programming problem in which coefficients of the objective function and some of constraints (the first kind constraints) are fuzzy interval numbers and remainder constraints (the second kind constraints) are expressed by fuzzy programming with symbol $\lessapprox$ for fuzzy inequality.

A fuzzy set on $\Re$ is called a fuzzy number if it is normal, convex and upper semi-continuous and its support set is compact. LR fuzzy number is a special fuzzy number used frequently. By an explanation of a fuzzy number, LR fuzzy number $\tilde{A}=(A^0,A^1,A^-,A^+)_{LR}$ is defined by the following
membership function:

$$\tilde{A}(x)=\begin{cases} L(\dfrac{A^0-x}{A^-}) & if \quad A^0-A^- \le x \le A^0 \\ 1 & if \quad A^0 < x \le A^1 \\ R(\dfrac{x\text{ - }A^1}{A^+}) & if \quad A^1 < x \le A^1+A^+, \end{cases}$$

where A^-, A^+ indicate the left and right spread respectively; $[A^0,A^1]$ denotes the peak of fuzzy number; $L,R:[0,1] \to [0,1]$ with $L(0)=R(0)=1$ and $L(1)=R(1)=0$ are strictly decreasing, continuous functions.

An inexact fuzzy interval linear programming model with fuzzy interval parameters and fuzzy inequalities thru symbol $\lessapprox$ is introduced by the following problem:

Problem A.1

$$\max\ \tilde{Z}^{\pm}=\sum_{j=1}^{n}\tilde{c}_j^{\pm}x_j^{\pm} \tag{A.1}$$

subject to

Constraints with fuzzy interval parameters:

$$\sum_{j=1}^{n}\tilde{a}_{ij}^{\pm}x_j^{\pm} \le \tilde{b}_i^{\pm}, \quad \forall i \in I_1 \tag{A.2}$$

Constraints with interval parameters and fuzzy inequality symbol:

$$\sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} \lessapprox b_i^{\pm}, \quad \forall i \in I_2 \tag{A.3}$$

Non-negativity constraints:

$$x_j^{\pm} \ge 0, j=1,2,...,n \tag{A.4}$$

where $I_1, I_2 \subseteq \{1,2,...,m\}$, $\tilde{c}_j^{\pm} = (c_j^-, c_j^+, \beta_j, \gamma_j)_{LR}$, $\tilde{a}_{ij}^{\pm} = (a_{ij}^-,\ a_{ij}^+, \beta_{ij}, \gamma_{ij})_{LR}$ and $\tilde{b}_i^{\pm} = (b_i^-, b_i^+, \eta_i, \lambda_i)$ are fuzzy interval numbers and $a_{ij}^{\pm}$, $b_i^{\pm}$ and $x_j^{\pm}$ are interval parameters /variables. An interval number has a known upper and lower bound but unknown distribution information (Ji et al., 2017). The objective function and the first part constraints of Problem A.1 include fuzzy interval parameters and hence have fuzzy interval properties. We will apply possibility measures to Problem A.1, which is fuzzy interval programming, for transforming it to interval programming.

As the Problem A.1 includes fuzzy interval parameters, we will consider a lower bound to the objective function and possibility measure to the constraints whose coefficients are fuzzy interval parameters for construction of possibility degree to these constraints. According to the possibility theory, we construct a fuzzy chance constrained programming (FCCP) method and apply it to maximize this lower bound subject to the constraints under fuzzy framework.

Based on extension principle in fuzzy framework, $\tilde{Z}^{\pm} \geq f^{\pm}$ and $\sum_{j=1}^{n} \tilde{a}_{ij}^{\pm} x_j^{\pm} \leq \tilde{b}_i^{\pm}$ are fuzzy interval events defined on possibility space $(\theta, P(\theta), Pos)$, whose possibility are:

$$\begin{aligned} &\mathrm{Pos}(\tilde{Z}^{\pm} \geq f^{\pm}) \\ &= \sup_{y_1, y_2 \in \Re} \{\min\{\mu_{\tilde{Z}^{\pm}}(y_1), \mu_{f^{\pm}}(y_2)\} \mid y_1 \geq y_2\} \end{aligned} \tag{A.5}$$

$$\begin{aligned} &\mathrm{Pos}(\sum_{j=1}^{n} \tilde{a}_{ij}^{\pm} x_j^{\pm} \leq \tilde{b}_i^{\pm}) \\ &= \sup_{y_1, y_2 \in \Re} \{\min\{\mu_{\tilde{a}_i^{\pm}}(y_1), \mu_{\tilde{b}_i^{\pm}}(y_2)\} \mid y_1 \geq y_2\} \end{aligned} \tag{A.6}$$

where $\sum_{j=1}^{n} \tilde{a}_{ij}^{\pm} x_j^{\pm} = \tilde{a}_i^{\pm}$ and *Pos* represents possibility. By using FCCP together with possibility measures, Problem A.1 is rewritten as follows:

Problem A.2

$$\max\ f^{\pm} \tag{A.7}$$

subject to

$$\mathrm{Pos}(\tilde{Z}^{\pm} \geq f^{\pm}) \geq \eta^{\pm}, \tag{A.8}$$

$$\mathrm{Pos}(\sum_{j=1}^{n} \tilde{a}_{ij}^{\pm} x_j^{\pm} \leq \tilde{b}_i^{\pm}) \geq h^{\pm}, \quad \forall i \in I_1 \tag{A.9}$$

$$\sum_{j=1}^{n} a_{ij}^{\pm} x_j^{\pm} \lessapprox b_i^{\pm}, \quad \forall i \in I_2 \tag{A.10}$$

$$x_j^{\pm} \geq 0, j = 1,2,...,n, \tag{A.11}$$

where $\eta^{\pm}$ and $h^{\pm}$ are defined as permitted possibility levels by the DMs. Now, we obtain a theorem to convert constraints (A.8) and (A.9) into parametric linear constraints.

Theorem A.1:

For any decision vector, it holds that:

$$\begin{aligned} &\mathrm{Pos}(\tilde{Z}^{\pm} \geq f^{\pm}) \geq \eta^{\pm} \\ &\Leftrightarrow \sum_{j=1}^{n} (c_j^{+} + \gamma_j R^{*}(\eta^{\pm})) x_j^{\pm} \geq f^{\pm} \end{aligned} \tag{A.12}$$

$$\operatorname{Pos}(\sum_{j=1}^{n}\tilde{a}_{ij}^{\pm}x_j^{\pm} \le \tilde{b}_i^{\pm}) \ge h^{\pm}$$ (A.13)

$$\Leftrightarrow \sum_{j=1}^{n}(a_{ij}^{-} - \beta_{ij}L^{*}(h^{\pm}))x_j^{\pm} \le b_i^{+} + \lambda_i R^{*}(h^{\pm})$$

where L^* and R^* are pseudo inverse functions defined as $L^*(\lambda) = \sup\{t \mid L(t) \ge \lambda\}$ and $R^*(\lambda) = \sup\{t \mid R(t) \ge \lambda\}$. For more study about this theorem and for its proof, the reader can be referred to the recent study (Nematian, 2015) about fuzzy random programming method.

By applying theorem A.1, we can easily rewrite Problem A.2 as follows:

Problem A.3

$$\max\ Z^{\pm} = \sum_{j=1}^{n}(c_j^{+} + \gamma_j R^{*}(\eta^{\pm}))x_j^{\pm}$$ (A.14)

subject to

$$\sum_{j=1}^{n}(a_{ij}^{-} - \beta_{ij}L^{*}(h^{\pm}))x_j^{\pm} \le b_i^{+} + \lambda_i R^{*}(h^{\pm}), \quad \forall i \in I_1$$ (A.15)

$$\sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} \lessapprox b_i^{\pm}, \quad \forall i \in I_2$$ (A.16)

$$x_j^{\pm} \ge 0, j = 1, 2, \ldots, n.$$ (A.17)

Now we use Zimmerman fuzzy approach (Zimmerman, 1978) to this problem, which is a fuzzy interval linear programming problem, for converting it to a deterministic problem.

We create the following membership functions for the objective function and relation (A.16) of Problem A.3.

$$\mu_{Z^{\pm}}(X) = \begin{cases} 0 & \text{if } Z^{\pm} \le f_d^{-} \\ \dfrac{Z^{\pm} - f_d^{-}}{f_d^{+} - f_d^{-}} & \text{if } f_d^{-} < Z^{\pm} \le f_d^{+} \\ 1 & \text{if } Z^{\pm} > f_d^{+} \end{cases}$$ (A.18)

$$\mu_{\sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm}}(X) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} \le b_i^{-} \\ \dfrac{b_i^{+} - \sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm}}{b_i^{+} - b_i^{-}} & \text{if } b_i^{-} < \sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} \le b_i^{+} \\ 0 & \text{if } \sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} > b_i^{+} \end{cases}$$ (A.19)

According to the Zimmerman fuzzy programming approach, let us assume that $\lambda^{\pm}$ corresponds to the membership functions of the fuzzy objective and/or constraints. Furthermore, DMs can establish the lower and upper bounds for the objective function they want to achieve, denoted by f_d^{-} and f_d^{+} respectively.

By considering the above membership functions, and Bellman and Zadeh's max-min operator, Problem A.3 will be transformed to the flowing interval linear programming problem:

Problem A.4

$$\max \ \lambda^{\pm} \tag{A.20}$$

subject to

$$\sum_{j=1}^{n}(c_j^{+}+\gamma_j R^{*}(\eta^{\pm}))x_j^{\pm} \geq f_d^{-}+\lambda^{\pm}(f_d^{+}-f_d^{-}), \tag{A.21}$$

$$\sum_{j=1}^{n}(a_{ij}^{-}-\beta_{ij}L^{*}(h^{\pm}))x_j^{\pm} \leq b_i^{+}+\lambda_i R^{*}(h^{\pm}), \quad \forall i \in I_1 \tag{A.22}$$

$$\sum_{j=1}^{n}a_{ij}^{\pm}x_j^{\pm} \leq b_i^{-}+(1-\lambda^{\pm})(b_i^{+}-b_i^{-}), \quad \forall i \in I_2 \tag{A.23}$$

$$0 \leq \lambda^{\pm} \leq 1, \ x_j^{\pm} \geq 0, j=1,2,\ldots,n. \tag{A.24}$$

where $\lambda^{\pm}$ is the control variable matching to the satis-faction degree for the fuzzy objective and/or constrains. Indeed, $\lambda^{\pm}$ denotes the satisfaction degree of the fuzzy constraints and is a variable between zero and one. Furthermore, f_d^{-} and f_d^{+} are the lower and upper bounds for the objective function respectively.

Huang (1996) introduced interactive two-step method for the interval-programming problem. According to this method, a two-step algorithm will be established to solve the above interval linear programming problem by analysing of interrelationships between the objective function and constraints and between their parameters and variables. Indeed, according to Huang (1996) and Huang & Cao (2011), the problem is divided into two deterministic sub-models. In the first sub-model, the goal is to reach the upper bound of the objective-function value (f^{+}), so the parameters and variables are set in their correct bounds in a way that the upper bound for the objective function occurs. Conversely, the second sub-model is based on the lower bound of the objective function (f^{-}).

Definition A.1: An interval number $a^{\pm}=[a^{-},a^{+}]$ is non-negative (positive) if $a^{-}\geq 0$ ($a^{-}>0$) and is non-positive (negative) if $a^{+}\leq 0$ ($a^{+}<0$).

Define the index sets $A_1=\{j: c_j^{+}+\gamma_j R^{*}(\eta^{\pm})\geq 0\}$ and $A_2=\{j: c_j^{+}+\gamma_j R^{*}(\eta^{\pm})\leq 0\}$. We can formulate the first and second sub-models based on index sets A_1 and A_2. The first sub- model, which corresponds to the upper bound of the objective function value, can be formulated as follows:

Problem A.5

$$\max \ \lambda^{+} \tag{A.25}$$

subject to

$$\begin{aligned}&\sum_{j\in A_1}(c_j^{+}+\gamma_j R^{*}(\eta^{-}))x_j^{+}\\ &+\sum_{j\in A_2}(c_j^{+}+\gamma_j R^{*}(\eta^{-}))x_j^{-} \quad ,\\ &\geq f_d^{-}+\lambda^{+}(f_d^{+}-f_d^{-})\end{aligned} \tag{A.26}$$

$$\begin{aligned}&\sum_{j\in A_1}\left|a_{ij}^{-}\right|^{-}\mathrm{Sign}(a_{ij}^{-})x_{ij}^{+}-L^{*}(h^{+})\sum_{j\in A_1}\beta_{ij}x_{ij}^{+}\\ &+\sum_{j\in A_2}\left|a_{ij}^{-}\right|^{+}\mathrm{Sign}(a_{ij}^{-})x_j^{-}-L^{*}(h^{-})\sum_{j\in A_2}\beta_{ij}x_j^{-}\\ &\leq b_i^{+}+\lambda_i R^{*}(h^{-}), \quad \forall i \in I_1\end{aligned} \tag{A.27}$$

(A.28)

$$\sum_{j\in A_1}\left|a_{ij}^{\pm}\right|^{-}\text{Sign}(a_{ij}^{\pm})x_j^{+}+\sum_{j\in A_2}\left|a_{ij}^{\pm}\right|^{+}\text{Sign}(a_{ij}^{\pm})x_j^{-}$$
$$\le b_i^{-}+(1-\lambda^{+})(b_i^{+}-b_i^{-}),\quad \forall i\in I_2$$

$$0\le\lambda^{+}\le 1,\ x_j^{+}\ge 0,\ j\in A_1,\ x_j^{-}\ge 0,\ j\in A_2. \tag{A.29}$$

Accordingly, the second sub-model corresponding to the lower bound of the objective function value is formulated as:
Problem A.6

$$\max\ \lambda^{-} \tag{A.30}$$

subject to

$$\begin{aligned}&\sum_{j\in A_1}(c_j^{+}+\gamma_j R^{*}(\eta^{+}))x_j^{-}\\&\sum_{j\in A_2}(c_j^{+}+\gamma_j R^{*}(\eta^{+}))x_j^{+}\ge,\\&+f_d^{-}+\lambda^{+}(f_d^{+}-f_d^{-})\end{aligned} \tag{A.31}$$

$$\begin{aligned}&\sum_{j\in A_1}\left|a_{ij}^{-}\right|^{+}\text{Sign}(a_{ij}^{-})x_{ij}^{-}-L^{*}(h^{-})\sum_{j\in A_1}\beta_{ij}x_{ij}^{-}+\\&\sum_{j\in A_2}\left|a_{ij}^{-}\right|^{-}\text{Sign}(a_{ij}^{-})x_j^{+}-L^{*}(h^{+})\sum_{j\in A_2}\beta_{ij}x_j^{+}\\&\le b_i^{+}+\lambda_i R^{*}(h^{+}),\quad\forall i\in I_1\end{aligned} \tag{A.32}$$

$$\begin{aligned}&\sum_{j\in A_1}\left|a_{ij}^{\pm}\right|^{+}\text{Sign}(a_{ij}^{\pm})x_j^{-}+\sum_{j\in A_2}\left|a_{ij}^{\pm}\right|^{-}\text{Sign}(a_{ij}^{\pm})x_j^{+}\\&\le b_i^{-}+(1-\lambda^{-})(b_i^{+}-b_i^{-}),\quad\forall i\in I_2\end{aligned} \tag{A.33}$$

$$\begin{aligned}&0\le\lambda^{-}\le\lambda_{opt}^{+},\\&x_j^{+}\ge x_{j\,\text{opt}}^{-},\ j\in A_2,\\&x_j^{-}\le x_{j\,\text{opt}}^{+},\ j\in A_1,\end{aligned} \tag{A.34}$$

where $\left|a_{ij}^{\pm}\right|^{+}=\begin{cases}a_{ij}^{+},\ \text{if}\ a_{ij}^{\pm}\ge 0\\-a_{ij}^{-},\ \text{if}\ a_{ij}^{\pm}<0\end{cases}$, $\left|a_{ij}^{\pm}\right|^{-}=\begin{cases}a_{ij}^{-},\ \text{if}\ a_{ij}^{\pm}\ge 0\\-a_{ij}^{+},\ \text{if}\ a_{ij}^{\pm}<0\end{cases}$ and $\text{Sign}(a_{ij}^{\pm})=\begin{cases}1,\ \text{if}\ a_{ij}^{\pm}\ge 0\\-1,\ \text{if}\ a_{ij}^{\pm}<0\end{cases}$.

Problems A.5 and A.6 are deterministic linear programming (LP) problems that can be solved by one of the LP solvers.
The optimal solutions of Problem A.1 are:

$$x_{j\,\text{opt}}^{\pm}=[x_{j\,\text{opt}}^{-},x_{j\,\text{opt}}^{+}],\quad\forall j \tag{A.35}$$

$$Z_{k\,\text{opt}}^{\pm}=[Z_{k\,\text{opt}}^{-},Z_{k\,\text{opt}}^{+}],\quad\forall k \tag{A.36}$$

where $x_{j\,\text{opt}}^{-}$ and $x_{j\,\text{opt}}^{+}$ are the optimal solutions, $Z_{k\,\text{opt}}^{-}$ and $Z_{k\,\text{opt}}^{+}$ are optimal objective function values, which are obtained from problems A.5 and A.6.

Supplementary Note B:

An algorithm is summarized for solving the problem discussed in section 2.2:

Algorithm 1

Data Entry:

Step 0. Define fuzzy parameters of problem 4 by using information of experts or DMs.

Model structure:

Step 1. Apply the FCCP method based on possibility measures and Zimmermann fuzzy approach:

-Convert problem 4 to problem 5.

Step 2. According to Huang (1996):

-Convert problem 5 to sub-problems 6 and 7.

Solution Procedure:

Step 3. Solve the obtained MIP problems by one of the MIP solvers. Then, the optimal solution of problem 4 is obtained by $S^{\pm}_{ijk\,opt} = [S^{-}_{ijk\,opt}, S^{+}_{ijk\,opt}]$, $A^{\pm}_{ijk\,opt} = q^{\pm}_{ij\,opt} - S^{\pm}_{ijk\,opt}$ and $q^{\pm}_{ij\,opt} = q^{-}_{ij} + \Delta q_{ij} z_{ij\,opt}$.

Table S1. Related economic data (Rial/m^3) and promised target of water allocation quantity for crops in Ajabshir Qaleh Chay dam

Crops (j)	Promised target of water allocation (in 10^3m^3) ($q_{ij}^{\pm}$)			Net benefit for water allocation ($NB_j^{\pm}$)	Shortage cost ($SC_j^{\pm}$)
	$i = 1$	$i = 2$	$i = 3$		
Wheat	[2.5, 3.8]	[1.7, 2.5]	[1.7, 2.5]	[2217, 2710]	[1713, 2093]
Barley	[2.5, 3.8]	[1.7, 2.5]	[1.7, 2.5]	[1785, 2182]	[1370, 1675]
Potato	[5.0, 7.5]	[5.0, 7.5]	[4.2, 6.3]	[3928, 4801]	[2728, 3335]
Onion	[5.0, 7.5]	[5.0, 7.5]	[4.2, 6.3]	[4697, 5741]	[3664, 4479]
Grape	[5.0, 7.5]	[5.0, 7.5]	[4.2, 6.3]	[2264, 2767]	[2597, 3174]
Walnut	[6.7, 10]	[5.8, 8.8]	[5.0, 7.5]	[38532, 47094]	[5426, 6241]
Almond	[5.8, 8.8]	[5.0, 7.5]	[4.2, 6.3]	[47397, 57930]	[6449, 7882]
Apple	[5.8, 8.8]	[5.0, 7.5]	[4.2, 6.3]	[1184, 1447]	[6371, 7656]

Table S2. Related economic data (Rial/m^3) and maximum allowable allocation amount from river for crops in Ajabshir Qaleh Chay dam

Crops (j)	maximum allowable water allocation (in 10^3m^3) ($q_{ijmax}^{\pm}$)			Allocation cost ($AC_{ij}^{\pm}$)			Transportation cost from river i ($TC_i^{\pm}$)		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
Wheat	17	11	10	[10.28, 10.47]	[6.85, 20.93]	[10.37, 12.56]			
Barley	17	11	10	[7.13, 10.05]	[13.7, 16.75]	[7.28, 8.38]			
Potato	17	11	10	[8.57, 16.68]	[8.22, 33.35]	[13.64, 20.01]			
Onion	17	11	10	[10.28, 26.86]	[6.85, 44.79]	[16.37, 22.40]	[22.17, 27.10]	[17.85, 21.82]	[39.28, 48.01]
Grape	17	11	10	[7.13, 15.87]	[13.70, 31.74]	[17.28, 19.04]			
Walnut	17	11	10	[8.57, 37.45]	[8.22, 62.41]	[13.64, 31.21]			
Almond	17	11	10	[10.28, 39.41]	[6.85, 78.82]	[16.37, 47.29]			
Apple	17	11	10	[17.13, 45.94]	[13.70, 76.56]	[27.28, 38.28]			

Table S3. Cost of increasing 1m^3 of water and available amount of water (in 10^3m^3) from other alternatives in Ajabshir Qaleh Chay dam

Crops (j)	$E_{jl}^{\pm}$			$\Delta T_{jl}^{\pm}$		
	$l = 1$	$l = 2$	$l = 3$	$l = 1$	$l = 2$	$l = 3$
Wheat	[1028, 1047]	[1713, 2093]	[857, 1256]	[9, 12]	[9, 11]	[10, 13]
Barley	[685, 1005]	[1370, 1675]	[822, 838]	[9, 12]	[9, 11]	[10, 13]
Potato	[1637, 1668]	[2728, 3335]	[1364, 2001]	[25, 27]	[22, 24]	[26, 28]
Onion	[1832, 2687]	[3664, 4479]	[2198, 2240]	[21, 23]	[27, 29]	[27, 29]
Grape	[1558, 1587]	[2597, 3174]	[1298, 1904]	[20, 22]	[27, 29]	[27, 29]
Walnut	[2713, 3745]	[5426, 6241]	[3256, 3121]	[25, 28]	[34, 36]	[34, 36]
Almond	[3869, 3941]	[6449, 7882]	[3225, 4729]	[28, 32]	[26, 30]	[25, 29]
Apple	[3186, 4594]	[6371, 7656]	[3823, 3828]	[28, 32]	[26, 29]	[25, 29]

Table S4. Maximum capacities of canal (per hectare) from river 1 to the farm planted with crop j (10^3m^3)

Capacity of canal ($c_{ij}^{\pm}$)								
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
$i=1$	3	4	3	4	3	4	3	4